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Spiral self-avoiding walks on the triangular lattice

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Abstract. We study the behaviour of spiral self-avoiding walks on the triangular lattice. The spiral constraint simply says that no step in an anticlockwise direction may be taken. Imposing the additional constraint that steps may not deviate from the straight ahead direction by $+\pi/3$ defines model I, previously solved by Joyce and Brak as well as by Lin. If deviations of $+2\pi/3$ are forbidden, we refer to this as model II, while model III allows deviations of both $+\pi/3$ and $+2\pi/3$. We find for both model II and model III that the number of n-step self-avoiding spirals is

$$s_n \sim c \exp(2\pi\sqrt{n}) \log(n/12)/n^{13/4}$$

where $c = \phi^2/768\gamma^5$, $\phi(\text{model II}) \approx 0.009$, $\phi(\text{model III}) \approx 0.16$ and $\gamma = 1 - 12(\log 2/\pi)^2$. The confluent logarithm is an additional feature not present in the simpler case of the square lattice and model I triangular spiral self-avoiding walks.

We make use of two new results in the theory of partitions.

1. Introduction

Since the introduction of spiral self-avoiding walks (SSAW) on the square lattice by Privman (1983) there has been considerable interest in generalisations of the model to other lattices and other dimensions. The two principal quantities of interest are the number of n-step walks s_n, and the mean end-to-end distance $\langle R_n \rangle$.

For the square lattice SSAW, the quantity s_n was determined to leading order by Guttmann and Wormald (1984) and Blöte and Hilhorst (1984). Guttmann and Hirschhorn (1984) gave the next-to-leading term for s_n, while the complete asymptotic expansion was given by Joyce (1984). Blöte and Hilhorst also showed that $\langle R_n \rangle \sim (3n)^{1/2}(\log n)/2\pi$ for square lattice SSAW.

For the triangular lattice, there are three distinct generalisations of the square lattice problem, which we refer to as models I, II and III. These are illustrated in figure 1(a). In model I the spiral constraint permits a step straight ahead or with an acute included angle of $\pi/3$. Model II permits a step straight ahead or with an obtuse included angle of $2\pi/3$, while model III permits a step straight ahead or through $+\pi/3$ or through $+2\pi/3$. Additional to this constraint is of course the global self-avoiding constraint.

Compared to models II and III, model I is relatively straightforward, being expressable, and indeed solvable, in much the same manner as the square lattice SSAW. Indeed, Joyce and Brak (1985) obtained the complete asymptotic expansion for s_n for

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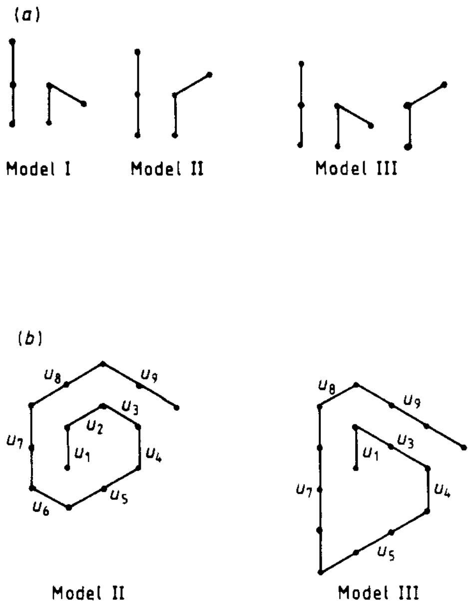

Figure 1. (*a*) The three possible triangular lattice models. (*b*) The labelling scheme for typical model II and model III spirals.

model I, while Lin (1985) independently obtained the leading term for s_n. Subsequently Lin and Liu (1986) provided an alternative, and somewhat simpler, derivation of Joyce and Brak's result and in another paper Liu and Lin (1985) obtained the mean square end-to-end distance for model I, which was found to be $\langle R_n \rangle \sim (6n)^{1/2}(\log n)/2\pi$.

In this paper we have studied models II and III, which require substantially different techniques for their solution. For all models considered, the method of analysis consists primarily of solving the problem of a single spiral, the number of which we denote s_n^*, and then concatenating two such spirals in order to obtain s_n. For the square lattice and model I on the triangular lattice the problem of single spirals can be simply related to the number of partitions of the integers, suitably restricted. No such identification is possible for models II and III as we shall show. Indeed, our analysis gives s_n only up to a multiplicative constant, which we estimate numerically.

Two aspects of our results deserve comment. Firstly, model III presents little additional difficulty over model II, while model II, despite a superficial similarity to model I is a totally different, and much more difficult, problem. Secondly, our earlier attempts at solving these problems were guided by series expansions. That is, we obtained 30–50 terms of various series for s_n^* and s_n and analysed these assuming a similar functional form to that obtained for the square and model I SSAW. Our series results turned out to be positively misleading, and provide a salutory lesson—if one were needed—of the difficulties of predicting unknown behaviour from series expansions. Finally we note a generalisation to three dimensions which has been introduced by Guttmann and Wallace (1985). They consider SSAW on the simple cubic lattice, and while mindful of the misleading series results for model II and III, tentatively conclude that $s_n \sim \mu^n n^{\gamma-1}$ and $\langle R_n^2 \rangle \sim An^{2\nu}$ with $\mu \approx 2.6560$, $\gamma \approx 1.24$ and $\nu \approx 0.655$. That is, similar behaviour to that found for ordinary three-dimensional SAW albeit with different critical exponents.

For all the two-dimensional problems mentioned, the spiral constraint completely changes the form of s_n from that of ordinary SAW, while it appears that the (less restrictive) three-dimensional spiral constraint does not cause such gross changes in functional form.

In the next section we derive the result for models II and III single spirals. In § 3 we discuss the concatenation of single spirals to generate full spirals and § 4 comprises a discussion and conclusion.

2. Determination of s_n^*

In figure 1(b) we show the labelling scheme used for model II and model III single spirals. Labelling the first segment u_1, successive segments in model II are labelled $u_2, u_3, u_4, \ldots$, as shown. For model III, since segments corresponding to turns through an included angle of $\pi/3$ may be missing, so too may some of the u_n. Thus the model III spiral shown in figure 1(b) has no segments corresponding to u_2 and u_6. The spirals shown in figure 1(b) correspond to the labellings

$$u_1 = u_2 = u_3 = u_4 = 1 \qquad u_5 = 2 \qquad u_6 = 1 \qquad u_7 = u_8 = u_9 = 2 \qquad \text{(model II)}$$

and

$$u_1 = 1 \qquad u_3 = 2 \qquad u_4 = 1 \qquad u_5 = 3 \qquad u_7 = 4 \qquad u_8 = 1$$
$$u_9 = 3 \qquad \text{(model III).}$$

We denote the number of steps by n, and $n = \Sigma_{i=1}^{k} u_i$, where the last non-empty segment is labelled u_k.

For convenience we set $u_0 = 0$ and note that for model II we have

$$u_i > 0 \qquad i = 1, k \tag{2.1}$$

while for model III we have the slightly more cumbersome constraint

$$\left.\begin{array}{ll} u_i \geqslant 0 & u_k > 0 \\ u_i + u_{i-1} \geqslant 0 & \end{array}\right\} \qquad i = 1, 2, \ldots, k-1. \tag{2.2}$$

Now the key equation that defines a single spiral is

$$u_{i-1} + u_i < u_{i+2} + u_{i+3} \qquad 1 \leqslant i \leqslant k-3. \tag{2.3}$$

While this is not immediately obvious, a few examples quickly serve to demonstrate the correctness of the result. To proceed further, we write

$$t_i = u_i + u_{i-1} \qquad i = 1, 2, \ldots, k \tag{2.4}$$

so that

$$u_i = t_i - t_{i-1} + t_{i-2} - t_{i-3} \ldots + (-1)^{i+1} t_1 \qquad i = 1, 2, \ldots, k. \tag{2.5}$$

Conditions (2.1) and (2.3) become

$$0 < t_i < t_{i+3} \qquad 1 \leqslant i \leqslant k-3 \tag{2.6}$$

and

$$u_1 = t_i - t_{i-1} + t_{i-2} - t_{i-3} \ldots + (-1)^{i+1} t_1 > 0 \qquad 1 \leqslant i \leqslant k. \tag{2.7}$$

Condition (2.2) for model III walks translates to (2.7) with the $>$ sign replaced by $\geqslant$ for $1<i\leqslant k-1$.

The problem now is to count the number of distinct decompositions of n such that

$$n=\sum_{i=1}^{k} u_i=\sum_{j\geqslant 0} t_{k-2j} \qquad (t_i=0 \text{ for } i\leqslant 0) \tag{2.8}$$

with t_i satisfying (2.6) and (2.7). To satisfy (2.6), it is convenient to define

$$d_i=t_i-t_{i-3} \qquad i=1,k \tag{2.9}$$

so that

$$t_i=\sum_{j\geqslant 0} d_{i-3j} \qquad i=1,k \tag{2.10}$$

and

$$n=\sum_{j\geqslant 0} d_{k-3j}+\sum_{j\geqslant 0} d_{k-3j-2}+\sum_{j\geqslant 0} d_{k-3j-4}+\sum d_{k-3j-6}+\ldots \tag{2.11}$$

while condition (2.7) becomes

$$u_i=(d_i-d_{i-1}+d_{i-2})+(d_{i-6}-d_{i-7}+d_{i-8}) \\ +(d_{i-12}-d_{i-13}+d_{i-14})+\ldots>0 \qquad 1<i\leqslant k. \tag{2.12}$$

We will initially focus on the sum (2.11) while ignoring the constraint (2.12) and will subsequently correct for the effect of (2.12). We shall need two results from the theory of partitions which have an independent interest of their own.

Theorem 1. Let $q_r(m)$ denote the number of partitions of m into r unequal positive integer parts. Then for $\lambda=r-(2/\pi)\sqrt{3m}\log 2=O(m^{1/3})$ we have asymptotically for large m

$$q_r(m)=\frac{1}{4m(6\gamma)^{1/2}}\exp[\pi(m/3)^{1/2}]\exp[-\pi\lambda^2/2\gamma(3m)^{1/2}] \tag{2.13}$$

where $\gamma=1-12(\log 2/\pi)^2=0.415\,8391\ldots$.

Theorem 2. Let $Q_k(n)$ denote the number of these partitions of n into unequal parts in which k is the largest summand. Then for large n

$$Q_k(n)\simeq Q(n)\frac{\lambda}{\sqrt{n}}\exp[-2(3\lambda)^{1/2}/\pi] \tag{2.14}$$

where

$$Q(n)\simeq\frac{1}{4\times 3^{1/4}n^{3/4}}\exp[\pi(n/3)^{1/2}] \tag{2.15}$$

is the total number of partitions of n into unequal parts and λ is determined from

$$\frac{2}{\pi}(3n)^{1/2}\log\lambda=\frac{(3n)^{1/2}}{\pi}\log n-k. \tag{2.16}$$

Corollary. For almost all partitions of n into unequal parts the largest summand is

$$k=\frac{(3n)^{1/2}}{\pi}\log n+\mathrm{O}(\sqrt{n}\,W(n)) \tag{2.17}$$

where $W(n)$ diverges arbitrarily slowly.

Theorem 1 can be deduced from results of Szekeres (1951). The sharp maximum of $q_r(m)$ for fixed m occurs when the number of summands is in the neighbourhood of $2\log 2(3m)^{1/2}/\pi$ (see Szekeres (1951), theorem 3) and the formula itself is obtained from that paper by fairly straightforward calculation. As a check it is worth noticing that

$$\sum_r q_r(m)\simeq\frac{1}{4m(6\gamma)^{1/2}}\exp[\pi(m/3)^{1/2}]\int_{-\infty}^{\infty}\exp[-\pi\lambda^2/2\gamma(3m)^{1/2}]\,\mathrm{d}\lambda$$

$$=\frac{\exp[\pi(m/3)^{1/2}]}{4\times 3^{1/4}m^{3/4}}\simeq Q(m) \tag{2.18}$$

as required.

A related result to theorem 2 was proved by Erdös and Lehner (1941), who proved that for almost all partitions of n into unequal parts the number of summands less than $c\sqrt{n}$ $(c>0)$ is

$$\frac{2(3n)^{1/2}}{\pi}\log\{2/[1+\exp(-c\pi/\sqrt{3})]\}(1+\mathrm{o}(1)). \tag{2.19}$$

Unfortunately this result is not strong enough to deduce theorem 2 and neither are the numerous results of Erdös and Szalay (1981) or Szalay and Turán (1977a, b). However, using the generating function

$$F(t)=\prod_{\nu=1}^{k}(1+t^\nu)-\prod_{\nu=1}^{k-1}(1+t^\nu)=t^k\prod_{\nu=1}^{k-1}(1+t^\nu)=\sum_n Q_k(n)t^n \tag{2.20}$$

one can deduce (2.14) by the same methods used to derive the results in Szekeres (1951). Details will be published elsewhere (Szekeres 1987).

Returning to our calculation, we re-arrange the terms in the sum (2.11) to give

$$n=\sum_{i=0}^{5}n_i \tag{2.21}$$

where

$$n_0=\sum_{j\geq 0}d_{k-6j}+\sum_{j\geq 1}d_{k-6j}+\sum_{j\geq 2}d_{k-6j}+\ldots=d_k+2d_{k-6}+3d_{k-12}+4d_{k-18}+\ldots \tag{2.22}$$

$$\begin{aligned}
n_1&=d_{k-7}+2d_{k-13}+3d_{k-19}+4d_{k-25}+\ldots\\
n_2&=d_{k-2}+2d_{k-8}+3d_{k-14}+4d_{k-20}+\ldots\\
n_3&=d_{k-3}+2d_{k-9}+3d_{k-15}+4d_{k-21}+\ldots\\
n_4&=d_{k-4}+2d_{k-10}+3d_{k-16}+4d_{k-22}+\ldots\\
n_5&=d_{k-5}+2d_{k-11}+3d_{k-17}+4d_{k-23}+\ldots.
\end{aligned} \tag{2.23}$$

Note that each n_i is of the form $m_1+2m_2+3m_3+\ldots$. Denote by $q_r(m)$ the number of decompositions of m in the form $m=\Sigma_{i=1}^{r}im_i$, $m_i>0$. This is of course equal to the number of partitions of m into r distinct summands, as can readily be seen from a Ferrers graph of a partition. Then, temporarily disregarding condition (2.12) and the fact that d_{k-1} does not appear in (2.21)-(2.23), we can write the number of decompositions of n satisfying (2.13) as $\hat{s}_n^*$, where the circumflex reminds us of the conditions that have been disregarded:

$$\hat{s}_n^* = \sum_{k>0} \sum_{n=\sum n_i} \prod_{i=0}^{5} q_{r_i}(n_i) \tag{2.24}$$

where

$$r_0 = \left[\frac{k+5}{6}\right] \qquad r_1 = \left[\frac{k-2}{6}\right] \qquad r_2 = \left[\frac{k+3}{6}\right]$$

$$r_3 = \left[\frac{k+2}{6}\right] \qquad r_4 = \left[\frac{k+1}{6}\right]$$

and

$$r_5 = \left[\frac{k}{6}\right].$$

Asymptotically, as n and r get large, we have $q_r(n) \sim q_{r+1}(n) \sim \max_r q_r(n)$ so with $r = [k/6]$, (2.24) simplifies to

$$\hat{s}_n^* = \sum_{k\geq 1} \sum_{n=\sum n_i} \prod_{i=0}^{5} q_r(n_i). \tag{2.25}$$

Now using (2.13) we can write

$$\hat{s}_n^* \simeq \sum_{r\geq 1} \sum_{n=\sum n_i} [4(6\gamma)^{1/2}]^{-6} \prod_{i=0}^{5} n_i^{-1} \exp[(\pi/\sqrt{3})$$
$$\times (\sqrt{n_0} + \sqrt{n_1} + \ldots + \sqrt{n_5})] \exp\left(\frac{-\pi}{2\sqrt{3}\gamma} \sum_{i=0}^{5} \lambda_i^2/\sqrt{n_i}\right) \tag{2.26}$$

where $\lambda_i = r - 2(3n_i)^{1/2} \log 2/\pi$. The terms in this formidable sum have a sharp maximum when $n_0 \approx n_1 \approx n_2 \approx n_3 \approx n_4 \approx n_5 = n/6$, and in that neighbourhood we can set $\lambda_j/\sqrt{n_j} = \sqrt{6}\lambda/\sqrt{n}$, and the sum over r is performed by integrating over λ, so that

$$\hat{s}_n^* \simeq \frac{3^{5/2}}{2^{37/4}\gamma^{5/2} n^{23/4}} \sum_{n=\sum n_i} \exp(\tfrac{1}{3}\pi(\sqrt{n_0} + \sqrt{n_1} + \ldots + \sqrt{n_5})). \tag{2.27}$$

To perform this final sum we set $n_i = n/6 + f_i$ so that

$$\exp(\pi/\sqrt{3} \textstyle\sum \sqrt{n_i}) = \exp[\pi(2n)^{1/2}] \exp\left(-\frac{3\pi}{\sqrt{2}n^{3/2}} \sum_{0\leq i\leq j\leq 4} f_i f_j\right) \tag{2.28}$$

and replacing the sum by a five-dimensional integral WRT f_0, f_1, f_2, f_3, f_4 we obtain

$$\hat{s}_n^* \sim \frac{\exp(\pi\sqrt{2n})}{64\sqrt{3}\gamma^{5/2} n^2}. \tag{2.29}$$

To pass from here to s_n^* we consider the effect of the two constraints we have neglected. Firstly, the fact that d_{k-1} does not appear in (2.23) means that the only constraint on d_{k-1} comes from (2.12). This gives two distinct restrictions:

$$d_{k-1} \leqslant (d_k + d_{k-6} + \ldots + d_{k-6n} + \ldots) - (d_{k-7} + d_{k-13} + d_{k-19} + \ldots)$$
$$+ (d_{k-2} + d_{k-8} + d_{k-14} + \ldots) \tag{2.30}$$
$$d_{k-1} \geqslant (d_{k-2} + d_{k-8} + d_{k-14} + \ldots) - (d_{k-3} + d_{k-9} + d_{k-15} + \ldots)$$
$$- (d_{k-7} + d_{k-13} + d_{k-19} + \ldots). \tag{2.31}$$

From (2.23) we see that $(d_k + d_{k-6} + \ldots + d_{k-6n} + \ldots)$ is the *maximum* summand in the unequal partition of $n_0 \sim n/6$. (A Ferrers graph makes this obvious.) From theorem 2 it follows immediately that (2.31) is irrelevant and can be replaced by $d_{k-1} > 0$ for almost all partitions. On the other hand (2.30) tells us that for almost all partitions d_{k-1} is only constrained by the requirement that it be less than the maximum summand in unequal partitions of n_0, i.e.

$$d_{k-1} \leqslant \frac{\sqrt{3}}{\pi}\left(\frac{n}{6}\right)^{1/2} \log\left(\frac{n}{6}\right) + \mathrm{O}(\sqrt{n}W(n)) \tag{2.32}$$

by the corollary of theorem 2. Hence the number of free choices for d_{k-1} is (for large n and for almost all spirals) $\sqrt{n}\log(n/6)/(\pi\sqrt{2})$. Thus $\hat{s}_n^*$ given by (2.29) must be multiplied by this expression to account for the freedom of d_{k-1}.

The final factor that must be introduced is the effect of constraint (2.12) for $i \neq k-1$. To account for this constraint we first observe that

$$q_{r-1}(m - m_1 r) = q_r(m)/2^{m_1}. \tag{2.33}$$

The left-hand side of (2.33) is the number of decompositions of large m into r distinct summands with m_1 fixed, where $m = m_r + 2m_{r-1} + \ldots + rm_1$. So from (2.33) we see that about 2^{-j} of all such decompositions have $m_1 = j$. This follows from the asymptotic form for $q_r(m)$ given by theorem 1. Similarly, for m_1 and m_2 small compared to (large) m, about $2^{-m_1-m_2}$ decompositions have these given values of m_1 and m_2. This result obviously generalises and we see that the effect of the constraint (2.12) on (2.29) is to introduce a factor

$$\phi = \sum 2^{-(d_1+d_2+\ldots+d_r)} \tag{2.34}$$

which is summed over all $d_i > 0$ satisfying (2.12). We have been unable to determine ϕ analytically, but by ordering the individual constraints (2.12) we have estimated ϕ numerically. That is, we consider the sequence of inequalities

$$\begin{array}{ll}
d_2 > d_1 & m = 2 \\
d_3 > d_2 - d_1 & m = 3 \\
d_4 > d_3 - d_2 & m = 4 \\
d_5 > d_4 - d_3 & m = 5 \\
d_6 > d_5 - d_4 & m = 6 \\
d_7 > d_6 - d_5 + d_1 & m = 7 \\
d_8 > d_7 - d_6 + d_2 - d_1 & m = 8 \\
\vdots & \vdots
\end{array}$$

(for model III the $>$ sign is replaced by $\geqslant$). Introduction of successive inequalities (2.35), labelled by m, gives a monotonic decreasing sequence of estimates for ϕ. These

Table 1. Estimates of ϕ (2.34) for model II and model III.

m	Model II	Model III
3	0.111 1111	0.555 556
4	0.079 3651	0.492 063
5	0.068 7831	0.433 862
6	0.053 1619	0.381 204
7	0.047 1679	0.361 604
8	0.042 6197	0.330 92
10	0.035 721	0.297 13
12	0.031 248	0.270 24
∞	$\approx$0.009	$\approx$0.16

are shown in table 1. Extrapolation of these against $1/m$ is quite linear and allows us to estimate $\phi_{\text{II}} \approx 0.009$ (model II) and $\phi_{\text{III}} \approx 0.16$ (model III).

Thus for model II we obtain

$$s_n^* = \frac{\phi_{\text{II}} \log(n/6) \exp[\pi(2n)^{1/2}]}{64\sqrt{6}\pi\gamma^{5/2} n^{3/2}} \approx 0.000\,16 \frac{\exp[\pi(2n)^{1/2}] \log(n/6)}{n^{3/2}} \tag{2.35}$$

while for model III only the constant changes to 0.0030. In the next section we show how single spirals can be concatenated to give full spirals.

3. Full spirals s_n

We consider the concatenation of two spirals to produce 'full' spirals. The concatenation of two single spirals for model II is as shown in figure 2, in which a spiral $S(m)$ is concatenated with a spiral $S'(m')$, the two spirals being of length m and m' steps respectively. As is true for the square lattice and model I triangular lattice full spirals, a necessary and sufficient condition for the existence of a full spiral is a *cutting line*, parallel to one of the lattice axes which cuts only one bond of the full spiral, no matter how far the cutting line is extended (see figure 2). For certain full spirals two cutting lines exist. In our formulation below these will be doubly counted, but as the number of such spirals is asymptotically negligible (proved explicitly by Guttmann and Wormald (1984) for the square lattice case), our result for s_n will be correct to leading order.

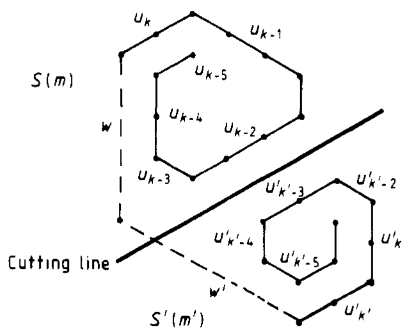

Figure 2. The concatenation of two single (model II) spirals $S(m)$ and $S'(m')$ to form a full spiral.

Referring again to figure 2, we concatenate $S(m)$ with $S'(m')$ by means of $w = u_{k+1} = u'_{k'+2}$ and $w' = u'_{k'+1} = u_{k'+2}$ where the labellings are explicitly shown in figure 2. The following conditions must be satisfied:

$$w + u_k > u_{k-2} + u_{k-3} \tag{3.1}$$

$$w' + u'_{k'} > u'_{k'-2} + u'_{k'-3} \tag{3.2}$$

$$w + w' > u_{k-1} + u_{k-2} + u'_{k'-1} + u'_{k'-2}. \tag{3.3}$$

If, in addition,

$$w' + u'_{k'} > u'_{k'-2} + u'_{k'-3} + u_k + u_{k-1} \tag{3.4}$$

then there exists a second cutting line and double counting occurs, as discussed above.

The above formulation applies to both models II and III, the only distinction coming from (2.1) and (2.2), as with single spirals.

Referring now to (2.22), we denote by σ_i $(i = 0, 1, \ldots, 5)$ the maximum summand in n_i, so that, for example,

$$\sigma_0 = d_k + d_{k-6} + d_{k-12} + d_{k-18} + \ldots. \tag{3.5}$$

Similarly

$$d_{i-1} + \sigma_1 = d_{i-1} + d_{i-7} + d_{i-13} + \ldots \tag{3.6}$$

and so from (2.12), successively setting $i = k$, $k-1$, $k-2$, $k-3$ we obtain

$$\begin{aligned} u_k &= \sigma_0 + \sigma_2 - \sigma_1 - d_{k-1} \\ u_{k-1} &= d_{k-1} + \sigma_1 - \sigma_2 + \sigma_3 \\ u_{k-2} &= \sigma_2 - \sigma_3 + \sigma_4 \\ u_{k-3} &= \sigma_3 - \sigma_4 + \sigma_5. \end{aligned} \tag{3.7}$$

Now in the concatenation of $S(m)$ and $S'(m')$ we have

$$n = m + m' + w + w'. \tag{3.8}$$

Equations (3.1)-(3.3) and (3.8) become

$$w = \sigma_5 + \sigma_1 - \sigma_0 + d + r \qquad (r > 0) \tag{3.9}$$

$$w' = \sigma'_5 + \sigma'_1 + \sigma'_0 + d' + r' \qquad (r' > 0) \tag{3.10}$$

$$n = m + m' + \sigma_1 + \sigma'_1 + \sigma_5 + \sigma'_5 - \sigma_0 - \sigma'_0 + d + d' + r + r' \tag{3.11}$$

$$w + w' = \sigma_1 + \sigma_4 + \sigma'_1 + \sigma'_4 + t \qquad (t > d + d') \tag{3.12}$$

where for notational simplicity we have set $d = d_{k-1}$ and $d' = d'_{k'-1}$. From (3.9)-(3.12)

$$r + r' = \sigma_0 + \sigma_4 - \sigma_5 + \sigma'_0 + \sigma'_4 - \sigma'_5 + t - d - d' \tag{3.13}$$

$$m + m' = n - (\sigma_1 + \sigma_4 + \sigma'_1 + \sigma'_4) - t. \tag{3.14}$$

The problem of determining s_n now reduces to the problem of estimating the number of choices of d, d', r and r' for a given $S(m)$ and $S'(m')$, which has then to be summed over all $t > 0$ and all choices of summands σ_j.

To leading order we can assume that each σ_i can be replaced by its most likely value, so that

$$\sigma_i \sim \sigma \simeq \frac{\sqrt{n}}{2\pi} \log \frac{n}{12} \qquad (i = 0, 1, \ldots, 5) \tag{3.15}$$

for almost all partitions, and (3.13) becomes

$$r + r' = 2\sigma + t - d - d' \tag{3.16}$$

so the number of possible choices for r, r', d and d' is

$$\sum (2\sigma + t - d' - d) = \tfrac{1}{2} t(t-1)(2\sigma + 1) \approx \sigma t^2 \tag{3.17}$$

where the summation is over $d + d' < t$.

For fixed d, the total number of simple spirals of length m and given maximal summands σ_i $(i = 0, \ldots, 5)$ is given by (2.29), (2.34) and theorem 2 as

$$s_m^*(\boldsymbol{\sigma}) \simeq \frac{\phi \exp[\pi(2m)^{1/2}]}{64\sqrt{3}\gamma^{5/2} m^2} \prod_{j=0}^{5} \frac{\mu_j}{(m/6)^{1/2}} \exp(-2\sqrt{3}\mu_j/\pi) \tag{3.18}$$

where

$$\log \mu_j = \tfrac{1}{2}\log(m/6) - \pi\sigma_j/(2m)^{1/2}. \tag{3.19}$$

Thus we obtain

$$\begin{aligned} s_n = \sum_{t>0} \sum_{\{\sigma_i\}} & \frac{\phi^2 \sigma t^2}{3 \times 2^8 \gamma^5 n^4} \exp\{\pi[(2m)^{1/2} + (2m')^{1/2}]\} \\ & \times \prod_{j,j'=0}^{5} \frac{\mu_j \mu'_{j'}}{(mm'/36)^{1/2}} \exp\left(-\frac{2\sqrt{3}}{\pi}(\mu_j + \mu'_j)\right). \end{aligned} \tag{3.20}$$

To satisfy (3.14) we write

$$m = n/2 - \sigma_1 - \sigma_4 + \nu \qquad m' = n/2 - \sigma'_1 - \sigma'_4 - \nu - t \tag{3.21}$$

where, as we subsequently see, $\nu = \mathrm{O}(n^{7/8})$. Substituting (3.21) into (3.20), we obtain

$$\begin{aligned} s_n = & \frac{\phi^2 \exp(2\pi\sqrt{n})}{768\gamma^5 n^4} \frac{\sqrt{n}}{2\pi} \log\left(\frac{n}{12}\right) \int_0^\infty t^2 \exp(-\pi t/\sqrt{n})\, \mathrm{d}t \int_{-\infty}^{\infty} \exp(-\pi\nu^2/n^{3/2})\, \mathrm{d}\nu \\ & \times \left(\int_0^\infty \mathrm{d}\sigma \frac{u}{(m/6)^{1/2}} \exp(-\pi\sigma/\sqrt{n} - 2\sqrt{3}\mu/\pi) \right)^4 \end{aligned} \tag{3.22}$$

where the last integral arises from the fact that each of σ_1, σ_4, σ'_1 and σ'_4 in (3.21) gives an identical integral and that $\mathrm{d}\sigma_j = -(2m)^{1/2}\, \mathrm{d}\mu_j/(\pi\mu_j)$, so that the contribution from the integral over the remaining terms in the product in (3.20) is just unity. The first two integrals are trivial. The last requires the identity $-\pi\sigma_1/\sqrt{n} = \log \mu_1 - \frac{1}{2}\log(n/12)$ from (2.16) and gives $(\pi/\sqrt{n})^4$. Hence we finally obtain

$$s_n = \frac{\phi^2 \exp(2\pi\sqrt{n})}{768\gamma^5 n^{13/4}} \log\left(\frac{n}{12}\right) \tag{3.23}$$

where $\phi \approx 0.009$ for model II and $\phi \approx 0.16$ for model III.

Table 2. Summary of known results for spiral self-avoiding walks.

Lattice	s_n	$\langle R_n \rangle$
Square	$\dfrac{\pi n^{-7/4}\exp[2\pi(n/3)^{1/2}]}{2^2 3^{5/4}}$	$(3n)^{1/2}\log(n)/2\pi$
Triangular model I	$\dfrac{\pi n^{-5/4}\exp[\pi(2n/3)^{1/2}]}{2^{-1/4}3^{7/4}}$	$(6n)^{1/2}\log(n)/2\pi$
Triangular model II	$\dfrac{\phi^2 n^{-13/4}\exp(2\pi n^{1/2})}{768\gamma^5}\log\left(\dfrac{n}{12}\right)$	?
	$\phi\approx 0.009,\ \gamma=1-12(\log 2/\pi)^2$	
Triangular model III	$\dfrac{\phi^2 n^{-13/4}\exp(2\pi n^{1/2})}{768\gamma^5}\log(n/12)$	?
	$\phi\approx 0.16,\ \gamma=1-12(\log 2/\pi)^2$	
Simple cubic	$\mu^n n^{\gamma-1}$ with $\mu\approx 2.6560$	$\langle R_n^2\rangle\sim n^{2\nu}$
	$\gamma\approx 1.24$	$\nu\approx 0.655$

4. Discussion and conclusion

We summarise the known exact and numerical results for spiral self-avoiding walks in table 2. The interesting feature that appears in the solution to models II and III is the presence of a confluent logarithmic term. It is the presence of this term, augmented by the slowly converging amplitude term ϕ, defined by (2.34), that makes conventional series analysis so unreliable in this case.

We give in table 3 enumerations of both single spirals and full spirals for both models. Analysis of these series is rather misleading, in that while the primary exponential growth term $\exp(2\pi\sqrt{n})$ is readily confirmed, an incorrect exponent of the correction term is suggested. For s_n, a preliminary analysis suggested $n^{-17/4}$, while the exact result is $n^{-13/4}\log(n)$. This discrepancy is clearly due to the fact that we are far removed from asymptotia.

Given both the exponential growth term and the correction terms $n^{-13/4}\log(n/12)$, the amplitude obtained from our enumerations is moving towards the predicted value, but is still, at $n=40$, a factor of 5 too large in the case of model II, though only 20% too large in the case of model III.

We have not attempted to calculate $\langle R_n \rangle$ for this model, but presumably this should be possible by the same techniques used in calculating s_n.

In conclusion we have obtained a new exact solution for a restricted self-avoiding walk problem. The model is of limited physical interest, but is a fascinating addition to the small class of exactly solvable models, and prompted the proof of two new theorems in the theory of partitions.

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Table 3.

n	Model II	Model III	Model II	Model III
1	1	1	1	1
2	2	3	2	3
3	4	8	4	7
4	8	20	8	15
5	16	47	14	31
6	31	106	26	59
7	61	230	43	110
8	115	479	74	198
9	213	973	120	347
10	388	1 924	197	592
11	691	3 712	311	997
12	1 218	7 021	495	1 641
13	2 110	13 034	768	2 666
14	3 617	23 780	1 189	4 266
15	6 113	42 732	1 811	6 741
16	10 238	75 703	2 748	10 525
17	16 945	132 360	4 116	16 268
18	27 802	228 664	6 136	24 882
19	45 180	390 611	9 058	37 717
20	72 838	660 296	13 299	56 683
21	116 479	1 105 321	19 370	84 504
22	184 936	1 833 358	28 069	125 031
23	291 556	3 014 694	40 399	183 716
24	456 694	4 917 036	57 856	268 125
25	710 907	7 958 127	82 374	388 873
26	1 100 192	12 786 252	116 736	560 647
27	1 693 123	20 401 469	164 574	803 723
28	2 591 830	32 337 878	231 007	1 146 013
29	3 947 417	50 936 233	322 749	1 625 731
30	5 982 953	79 750 436	449 089	2 294 964
31	9 026 001	124 149 022	622 263	3 224 587
32	13 556 212	192 204 697	858 935	4 510 551
33	20 273 031	296 001 288	1181 048	6 282 294
34	30 193 155	453 548 269	1618 209	8 714 034
35	44 789 076	691 574 373	2209 299	12 039 318
36	66 186 878	1049 590 078		
37	97 446 176	1585 770 518		
38	142 956 659	2384 452 588		
39	208 997 567	3573 359 776		
40	304 524 831	5331 138 923		
41	442 276 098			
42	640 317 828			
	'Full' spirals on triangular lattice		Single spirals on triangular lattice	

References

Blöte H W J and Hilhorst H J 1984 *J Phys. A: Math. Gen.* **17** L111
Erdös P and Lehner J 1941 *Duke Math. J.* **8** 335–45
Erdös P and Szalay M 1981 *Colloq. Math. Soc. J. Bolyai* **34** 397–450
Guttmann A J and Hirschhorn M 1984 *J Phys A: Math. Gen.* **17** 3613
Guttmann A J and Wallace K C 1985 *J Phys. A: Math. Gen.* **18** L1049
Guttmann A J and Wormald N C 1984 *J. Phys. A: Math. Gen.* **17** L271

Joyce G S 1984 *J. Phys. A: Math. Gen.* **17** L463
Joyce G S and Brak R 1985 *J. Phys. A: Math. Gen.* **18** L293
Lin K Y 1985 *J. Phys. A: Math. Gen.* **18** L145
Lin K Y and Liu K C 1986 *J. Phys. A: Math Gen.* **19** 545
Liu K C and Lin K Y 1985 *J. Phys. A: Math. Gen.* **18** L647
Privman V 1983 *J. Phys. A: Math. Gen.* **16** L571
Szalay, M and Turán P 1977a *Acta Math. Acad. Sci. Hung.* **29** 361-79
—— 1977b *Acta Math. Acad. Sci. Hung.* **29** 381-92
—— 1978 *Acta Math. Acad. Sci. Hung.* **32** 129-55
Szekeres G 1951 *Q. J. Math.* **2** 85-108
—— 1987 *Bull. Aust. Math. Soc.* submitted